Worked solutions for FP1 Paper
January 2006

Multiply each entry by

1. (i) • $2B = 2\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ 2 & 8 \end{pmatrix}$

Note that this is DIFFERENT from $B^2 = B \times B$

"rows × columns"

• A is a 2×2 matrix. C is a 3×2 matrix.
You can only add or subtract two matrices if they have the same number of rows and columns.
Therefore $A + C$ is IMPOSSIBLE

• $CA = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$

$1 \times 3 + (-1) \times 2$ $1 \times 4 + (-1) \times 1$
$0 \times 3 + 2 \times 2$ $0 \times 4 + 2 \times 1$
$0 \times 3 + 1 \times 2$ $0 \times 4 + 1 \times 1$

3×2 2×2 3×2

These are the same!

A LOT OF PEOPLE IN THE EXAM THOUGHT CA WAS IMPOSSIBLE! REMEMBER: MATRIX MULTIPLICATION IS POSSIBLE IF THE NUMBER OF COLUMNS IN THE FIRST MATRIX IS THE SAME AS THE NUMBER OF ROWS IN THE SECOND. The two matrices don't necessarily have to have the same number of rows and columns!

• A and B are both 2×2 matrices and so $A - B$ is possible:
$A - B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 0 & -2 \end{pmatrix}$ (subtract corresponding entries)

(ii) We need to find two matrices X, Y such that $XY \neq YX$.
Although we have calculated CA, AC is impossible.

Although this is not given in the mark scheme, $AC \neq CA$ and this does show that matrix multiplication is not commutative!

We can compute both AB and BA:

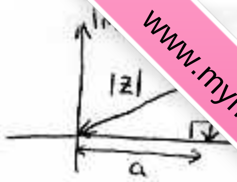
$AB = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 4 & 5 \end{pmatrix}$

$BA = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 8 & 11 \end{pmatrix}$

You LOST A MARK if you did not make this conclusion!

so $AB \neq BA$ showing that matrix multiplication is not commutative.

2. (i) $|z|$ is the MODULUS of z: $|z| = \sqrt{a^2+b^2}$



z^* is the COMPLEX CONJUGATE of z: $z = a - bj$ (just make the imaginary part the negative of what it was before).

(The question should have said that a, b are real numbers but this is certainly assumed)

(ii) $$zz^* = (a+bj)(a-bj) = a^2 - (bj)^2$$ $\boxed{\text{Difference of two squares!}}$

$$= a^2 - b^2j^2$$

$$= a^2 + b^2 \qquad \boxed{j^2 = -1}$$

$|z| = \sqrt{a^2+b^2}$ (in part (i)) so $|z|^2 = a^2 + b^2$

So $zz^* - |z|^2 = a^2+b^2 - (a^2+b^2) = 0$

3. $$\sum_{r=1}^{n} (r+1)(r-1) = \sum_{r=1}^{n} (r^2 - 1)$$

$\boxed{\text{NB. This is } \underline{\text{NOT}} \left(\sum_{r=1}^{n} r + 1\right)\left(\sum_{r=1}^{n} r - 1\right)!}$

You must first expand the brackets before you can use the formulae for $\sum_{r=1}^{n} r, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r^3$.

$$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} - n$$

$\boxed{\text{NB. } \sum_{r=1}^{n} 1 = n \text{ } \underline{\text{not}} \text{ } 1!}$

$$= \frac{n(n+1)(2n+1) - 6n}{6}$$

(Put over a common denominator)

$$= \frac{n[(n+1)(2n+1) - 6]}{6}$$

(Take out n as a common factor: Do NOT multiply out brackets until you have first taken out all common factors).

$$= \frac{n(2n^2 + 3n + 1 - 6)}{6}$$

$$= \frac{n(2n^2 + 3n - 5)}{6}$$

$$= \frac{n(2n+5)(n-1)}{6}$$

4. (i) Multiplying out the matrices on the left-hand side gives

$\begin{pmatrix} (6x - 2y) \\ (-3x + y) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. But this is true exactly if both $6x - 2y = a$ and $-3x + y = b$

(Just writing the equations down is fine!)

(ii) The determinant of $\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}$ is $6\times1-(-2)\times(-3)=6-6=0$

Whenever the determinant of a 2×2 matrix is zero, any corresponding of simultaneous equations EITHER has no solution OR has infinitely many solutions

[Whenever the determinant is zero the left-hand sides of the two equations are just a multiple of each other, e.g. here $6x-2y=-2(-3x+y)$.
So either the two equations are identical (when the right-hand sides are the same multiple, e.g. $_{\times(-2)}\left(\begin{array}{l} 6x-2y=-2 \\ -3x+y=1 \end{array}\right)_{\times(-2)}$)- so essentially you have just one equation of which there are infinitely many solutions - or you get two equations where all of the solutions of one equation do not satisfy the other,
e.g. $_{\times(-2)}\left(\begin{array}{l} 6x-2y=-3 \\ -3x+y=1 \end{array}\right)_{\times(-3)}$.

If the determinant is non-zero there is a unique solution, which is found using the inverse of the matrix (and the inverse exists when the determinant is non-zero).]

5. (i) The coefficient of x^3 is 1 so we don't have to first divide through the equation by some number.

(*) [The cubic equation with roots α, β, γ (and written so the coefficient of x^3 is 1) is $(x-\alpha)(x-\beta)(x-\gamma)=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$]

So if $x^3+3x^2-7x+1=0$

then $-(\alpha+\beta+\gamma)=3$ so $\alpha+\beta+\gamma=-3$

$\alpha\beta+\beta\gamma+\gamma\alpha=-7$

and $-\alpha\beta\gamma=1$ so $\alpha\beta\gamma=-1$

(ii) In (*), replacing α, β, γ with $2\alpha, 2\beta, 2\gamma$ it follows that we need to know:

$$2\alpha+2\beta+2\gamma=2(\alpha+\beta+\gamma)=2\times(-3)=-6$$

$$2\alpha\times2\beta+2\beta\times2\gamma+2\gamma\times2\alpha=4\alpha\beta+4\beta\gamma+4\gamma\alpha=4(\alpha\beta+\beta\gamma+\gamma\alpha)=4\times(-7)=-28$$

$$2\alpha\times2\beta\times2\gamma=8\alpha\beta\gamma=8\times(-1)=-8$$

so the cubic equation is $x^3-(-6)x^2+(-28)x-(-8)=0$

so $x^3+6x^2-28x+8=0$

You LOSE ONE MARK if you do not write "$\ldots\underline{\underline{=0}}$" as you are asked to give an EQUATION

Alternative method to 5.(ii):

$x^3+3x^2-7x+1=0$ when $x=\alpha, \beta$ or γ

If we put $w=2x$, so $x=\frac{w}{2}$

then $\left(\frac{w}{2}\right)^3+3\left(\frac{w}{2}\right)^2-7\left(\frac{w}{2}\right)+1=0$ when $\frac{w}{2}=\alpha, \beta$ or γ, i.e. $w=2\alpha, 2\beta$ or 2γ

so $\frac{w^3}{8}+\frac{3w^2}{4}-\frac{7w}{2}+1=0$

(This is fine, but we can make it a bit neater by multiplying both sides by 8...)

so $w^3+6w^2-28w+8=0$ is a cubic equation with roots $w=2\alpha, 2\beta, 2\gamma$.

6. When $n=1$ the statement is true since

$$\text{LHS}=\sum_{r=1}^{1}\frac{1}{r(r+1)}=\frac{1}{1\times 2}=\frac{1}{2} \text{ and RHS}=\frac{1}{1+1}=\frac{1}{2}=\text{LHS}$$

Assume the statement is true for $n=k$, k some positive integer:

i.e. assume $$\sum_{r=1}^{k}\frac{1}{r(r+1)}=\frac{k}{k+1}$$

Then

We want this to be $\frac{k+1}{k+2}$

$$\sum_{r=1}^{k+1}\frac{1}{r(r+1)}=\sum_{r=1}^{k}\frac{1}{r(r+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

Don't forget to write down all explanation!

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

so the statement is true for $n=k+1$ whenever it is true for $n=k$.

Since it is true for $n=1$, it is true for all positive integers n by the principle of induction.

(You MUST use induction for this question since the question specifically asks for it).

7. (i) $x^2 \geqslant 0$ and so $3 + x^2 \neq 0$ and therefore $y \neq 0$ for all rea

(ii) The vertical asymptotes are precisely when the denominator is

i.e. $4 - x^2 = (2+x)(2-x) = 0$ so there are two vertical asympto

$x = -2$ and $x = 2$.

The horizontal asymptote is $y = -1$

[It is fine just to write this down

for any curve "$y = \dfrac{\text{quadratic}}{\text{quadratic}}$" the horizontal asymptote is

"$y = \dfrac{\text{coefficient of } x^2 \text{ in numerator}}{\text{coefficient of } x^2 \text{ in denominator}}$"

This is because as x becomes very large (positive or negative) the x^2 terms become far more significant].

NB. the horizontal asymptote is only $y=0$ for "$y = \dfrac{\text{linear}}{\text{quadratic}}$"

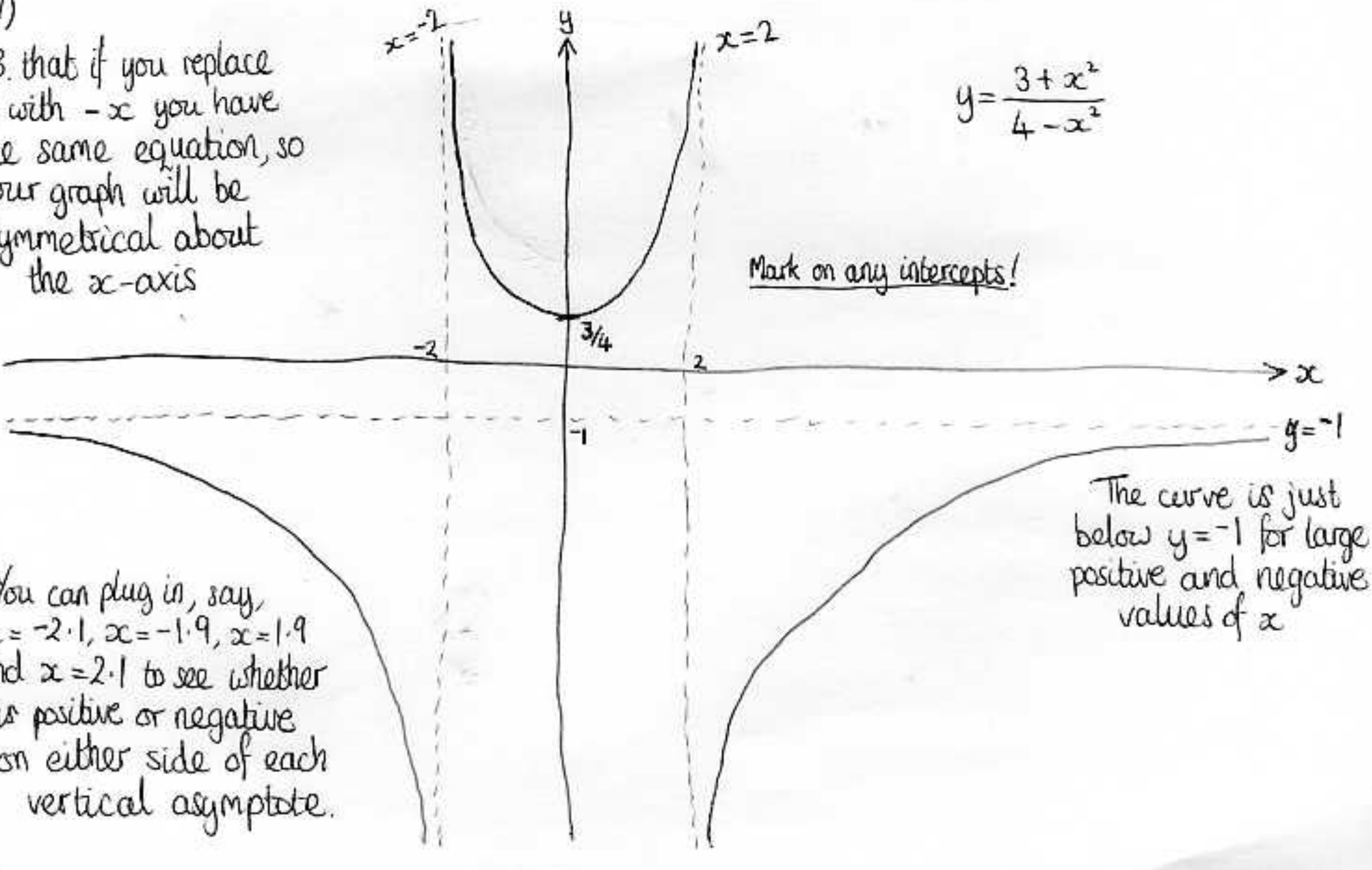
(iii) EITHER plug in a large positive value of x – say $x = 100$ – and a large negative value of x – say $x = -100$. In both cases you get a number which is just slightly less than -1. (You NEED TO SHOW THIS WORKING).
Therefore the curve is slightly BELOW the horizontal asymptote $y = -1$ when x is large and positive and when x is large and negative.

OR you could write $y = \dfrac{3+x^2}{4-x^2} = \dfrac{-(4-x^2)+7}{4-x^2} = -1 + \dfrac{7}{4-x^2}$

When x is large (positive or negative) $4 - x^2$ is a large negative number and $\dfrac{7}{4-x^2}$ is a small negative number,
so y is slightly less than -1.

(iv)

NB. that if you replace x with $-x$ you have the same equation, so your graph will be symmetrical about the x-axis

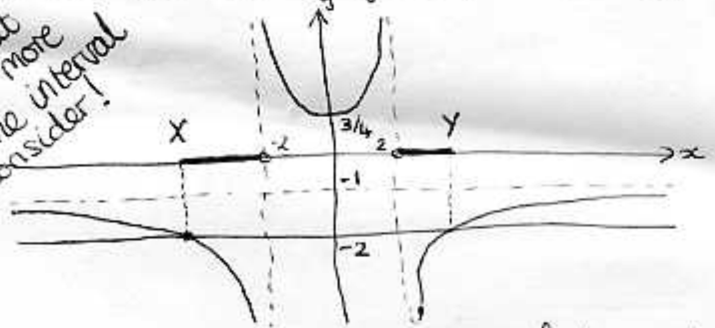


The curve is just below $y = -1$ for large positive and negative values of x

You can plug in, say, $x = -2{\cdot}1$, $x = -1{\cdot}9$, $x = 1{\cdot}9$ and $x = 2{\cdot}1$ to see whether y is positive or negative on either side of each vertical asymptote.

(v) From the graph, the values of x for which $\frac{3+x^2}{4-x^2} \leqslant -2$ are the the values of x for which the graph is on or below the line $y=-2$

NB that there is more than one interval to consider!



so $X \leqslant x < -2$ or $2 < x \leqslant Y$

[NB. that when $x=2$ or $x=-2$ the inequality is not true]

To find X and Y (by the symmetry of the graph, we know X should be $-Y$):

when $\frac{3+x^2}{4-x^2} = -2$, $\quad 3+x^2 = -2(4-x^2) = -8+2x^2$

so $\quad x^2 = 11$

and $x = \pm\sqrt{11}$ $\quad$ (so $X = -\sqrt{11}$ and $Y = \sqrt{11}$ in the picture above).

Therefore if $\frac{3+x^2}{4-x^2} \leqslant -2$ then either $-\sqrt{11} \leqslant x < -2$

or $\quad 2 < x \leqslant \sqrt{11}$

8. (i) $\alpha = 1+j$

so $\alpha^2 = (1+j)^2 = 1+2j+j^2 = 2j$ $\quad \{j^2=-1\}$

$\alpha^3 = \alpha^2\alpha = 2j(1+j) = 2j+2j^2 = -2+2j$

α satisfies $z^3+3z^2+pz+q=0$

so $\quad \alpha^3+3\alpha^2+p\alpha+q=0$

so $\quad (-2+2j)+3\times 2j+p(1+j)+q=0$

so $\quad (-2+p+q)+(8+p)j=0$

The real part and the imaginary part must both be zero.

so $-2+p+q=0$

and $\quad 8+p=0$

(NB. The question says that p, q are real numbers.)

so $p=-8$ and $-2-8+q=0$

so $q=10$

(ii) $\alpha = 1+j$ is a root so its complex conjugate, $1-j$, is also a root.

Recall the ideas in Question 5.

The sum of the roots is -3. $\quad$ {$-$(coefficient of z^2)}

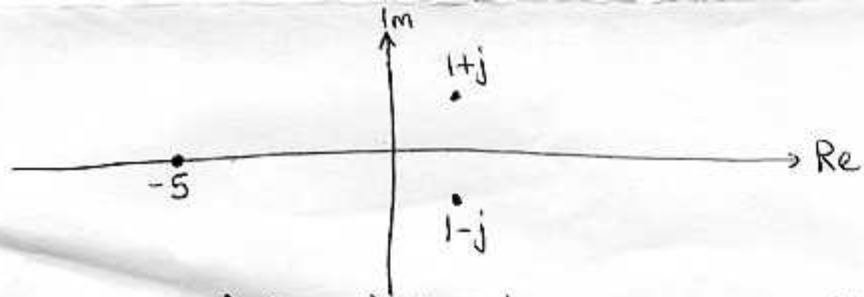
So if the third root is x then $(1+j)+(1-j)+x=-3$

so $\quad 2+x=-3$ and $x=-5$

So the third root is -5

(You could also say that the product of the roots is -10)

(iii)



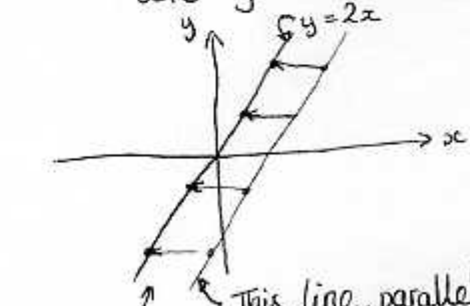
9\. (i) The image of $(10, 50)$ has the same y co-ordinate, 50, and lies on the line $y = 2x$. If $y = 50$ and $y = 2x$ then $50 = 2x$ so $x = 25$.
So the image of $(10, 50)$ is $(25, 50)$.

(ii) In the same way, the image of (x, y) has the same y co-ordinate, y, and lies on the line $y = 2x$, so $x = \frac{1}{2}y$.
So the image of (x, y) is $(\frac{1}{2}y, y)$.

(iii) By part (ii), you need to look for all points (x, y) so that $(\frac{1}{2}y, y) = (3, 6)$
This is all points (x, y) with $y = 6$. So the line l has equation $y = 6$.

(iv) Since the y co-ordinate doesn't change under the transformation T, for any line parallel to the x-axis all points on the line are mapped to a single point on the line $y = 2x$.

(v) If you consider any line parallel to $y = 2x$, all points on such a line are moved by the <u>same distance</u> parallel to the x-axis onto the line $y = 2x$, and the effect is that the line has been translated parallel to the x-axis.
So the answer is all lines parallel to $y = 2x$.



Each point on the line is translated through the same distance.

This line, parallel to $y = 2x$, is mapped onto $y = 2x$ by translation parallel to the x-axis.

If you're stuck here, just pick a line and see what happens to it under T... try to think which lines you must choose so that applying T will just move them parallel to the x-axis

(vi) Using part (ii), $(1, 0)$ is mapped to $(0, 0)$ ← This gives the 1st column of the matrix
and $(0, 1)$ is mapped to $(\frac{1}{2}, 1)$ ← This gives the 2nd column of the matrix
So the matrix is $\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$.

(vii) 'singular' means that 'the matrix has zero determinant'.
The determinant of $\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ is $0 \times 1 - \frac{1}{2} \times 0 = 0$

You could call the transformation "many-to-one"

Zero determinant means that the matrix, and therefore the transformation, has no inverse. To get full marks, the best explanation is to say that the reason the transformation has no inverse is that it maps infinitely many points to each single point on the line $y = 2x$.